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Effect of longitudinal surface curvature on boundary layers

By RODDAM NARASIMHA AND S. K. OJHA
Department of Aeronautical Engineering, Indian Institute of Science, Bangalore, India

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We consider here the higher order effect of moderate longitudinal surface curvature on steady, two-dimensional, incompressible laminar boundary layers. The basic partial differential equations for the problem, derived by the method of matched asymptotic expansions, are found to possess similarity solutions for a family of surface curvatures and pressure gradients. The similarity equations obtained by this anaylsis have been solved numerically on a computer, and show a definite decrease in skin friction when the surface has convex curvature in all cases including zero pressure gradient. Typical velocity profiles and some relevant boundary-layer characteristics are tabulated, and a critical comparison with previous work is given.

1. Introduction

In the classical boundary-layer theory of Prandtl, the surface curvature has no explicit effect on the boundary layer. It is of course crucial in determining the inviscid surface velocity distribution, to which the boundary-layer velocity profile itself should tend towards the outer edge of the layer. But once this distribution along the surface is known (either from inviscid theory or from experiment), the curvature of the surface is irrelevant to the solution of the classical boundary-layer problem. At lower Reynolds numbers, however, the curvature begins to influence the boundary layer explicitly, and it is this higher order effect that we want to study here. We confine ourselves to steady two-dimensional laminar incompressible motion.

The subject has been dealt with by several workers previously and has been rather controversial as their results do not agree. Thus while Tani (1949, 1954) and Murphy (1953, 1962, 1965) conclude that the skin friction will decrease on the convex side of the surface because of the curvature (a trend confirmed by the present work), Yen & Toba (1961, 1962) arrive at just the opposite conclusion. A systematic formulation of the general second-order boundary-layer theory, using singular perturbation techniques, has been presented by Van Dyke (1962), and used by him subsequently (1964) to discuss the flow past a parabolic cylinder. Here he finds a reduction of skin friction due to convex curvature, which is the same trend as Murphy's results show. However, no systematic general analysis of the kind of problem considered by Murphy seems to have been made yet and the purpose of this paper is precisely to provide such analysis, and to point out the reasons for the errors and discrepancies in previous work, especially that of Murphy and Yen & Toba.

In what follows it is assumed that the curvature K is only moderate, meaning essentially that its product with the boundary layer thickness δ is small. Any more general theory (for larger curvatures) would then have to reduce to the case considered by us in the limit. Such a more general theory does not seem to exist as yet, though claims have been made to include the effect of large curvatures (see § 5). It is easy to convince oneself that in most aeronautical applications the parameter $K\delta$ is indeed small. However, flight at high altitude involves lower Reynolds numbers and hence leads to larger values of $K\delta$.

In the following section we review the basic formulation of the problem as given by Van Dyke, using singular perturbation techniques. The partial differential equations so derived for the higher order problem are then reduced to an ordinary differential equation in § 3 by similarity analysis. It will be found here that similarity solutions are possible for a class of surface curvatures and pressure gradients which are essentially an extension of the classical Falkner–Skan family. In § 4 we present solutions of these equations, obtained numerically on a digital computer. These results are compared with other work in § 5, which includes a brief critique and discussion of the previous analyses.

2. Formulation of the problem

In this section we give a brief resumé of the derivation of the basic higher order equations, following the treatment of Van Dyke (1962). The Navier–Stokes equations for the steady two-dimensional motion of an incompressible viscous fluid can be written, in vector non-dimensional form, as

$$\operatorname{div} \mathbf{U} = 0, \tag{2.1}$$

$$\mathbf{U} . \operatorname{grad} \mathbf{U} = -\operatorname{grad} P - \epsilon^2 \operatorname{curl} \operatorname{curl} \mathbf{U}. \tag{2.2}$$

Here $\mathbf{U}$ is the fluid velocity divided by the velocity at upstream infinity (say U^*) and P is the pressure divided by ρU^{*2}; ρ is the density, and the space co-ordinates $\mathbf{X}$ are non-dimensionalized by some length L^* characteristic of the body in the flow. Finally we have written ϵ^2 for ν/U^*L^*, which is the reciprocal of the Reynolds number R, ν being the kinematic viscosity. In boundary-layer theory we are concerned with the limit $\epsilon \to 0$.

We take as our boundary conditions

$$\mathbf{U} \to \mathbf{U}^* \quad \text{at upstream infinity,} \tag{2.3a}$$

and

$$\mathbf{U} = 0 \quad \text{on the surface } S, \tag{2.3b}$$

where $\mathbf{U}^*$ is the (constant) velocity vector at infinity and S is the surface past which the fluid is flowing.

Following the usual procedure in the singular perturbation analysis, we first make an outer expansion of our variables in the limit $\mathbf{X}$ fixed, $\epsilon \to 0$; thus we write

$$\mathbf{U} = \mathbf{U}_0(\mathbf{X}) + \epsilon \mathbf{U}_1(\mathbf{X}) + \dots, \tag{2.4a}$$

$$P = P_0(\mathbf{X}) + \epsilon P_1(\mathbf{X}) + \dots. \tag{2.4b}$$

It can be shown that this outer flow is irrotational to all orders in ϵ, and hence can be obtained by solving a potential flow problem to satisfy the outer boundary condition (2.3a) and a suitable 'inner' boundary condition, to be determined by matching (see below).

To find an inner limit which will be valid near the surface, it is convenient first to introduce an orthogonal curvilinear co-ordinate system, made up of straight lines normal to S and curves parallel to S. We denote distance along the normal lines by Y, and distance along the other family by X. The inner limit is then defined by the process $\epsilon \to 0$ as $x \equiv X$ and $y \equiv Y/\epsilon$ are fixed, and the inner expansion of the variables is

$$u = u_0(x,y) + \epsilon u_1(x,y) + \ldots, \tag{2.5a}$$

$$v = \epsilon v_0(x,y) + \epsilon^2 v_1(x,y) + \ldots, \tag{2.5b}$$

$$p = p_0(x,y) + \epsilon p_1(x,y) + \ldots. \tag{2.5c}$$

(We use lower case symbols for inner variables and upper case symbols for outer variables.) The equations of motion then give

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0, \tag{2.6a}$$

$$\frac{\partial p_0}{\partial y} = 0, \tag{2.6b}$$

and

$$u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial y} = -\frac{dp_0}{dx} + \frac{\partial^2 u_0}{\partial y^2}, \tag{2.6c}$$

for the zeroth order quantities; these equations are just the familiar Prandtl boundary-layer equations, showing that curvature has no explicit influence to this order. The next order terms give

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = -K \frac{\partial}{\partial y}(u_0 y), \tag{2.7a}$$

$$\frac{\partial p_1}{\partial y} = K u_0^2, \tag{2.7b}$$

and

$$u_0 \frac{\partial u_1}{\partial x} + v_0 \frac{\partial u_1}{\partial y} + u_1 \frac{\partial u_0}{\partial x} + v_1 \frac{\partial u_0}{\partial y}$$
$$= -\frac{\partial p_1}{\partial x} + \frac{\partial^2 u_1}{\partial y^2} + K\left\{y\left(u_0 \frac{\partial u_0}{\partial x} + \frac{\partial p_0}{\partial x}\right) + \frac{\partial u_0}{\partial y} - u_0 v_0\right\}$$

where $K = K(x)$ is the surface curvature.

Matching the inner and outer solutions (Van Dyke 1962) gives us the following boundary conditions for the inner and outer equations:

$$V_0(X,0) = 0, \tag{2.8a}$$

$$u_0(x,0) = 0 = v_0(x,0), \tag{2.8b}$$

$$u_0(x,\infty) = U_0(X,0), \quad p_0(x) = P_0(X,0), \tag{2.8c}$$

$$V_1(X,0) = v_0(y) + \left(\frac{\partial U_0}{\partial X}\right)_{Y=0} y, \quad y \to \infty, \tag{2.8d}$$

$$u_1(x,0) = 0 = v_1(x,0), \tag{2.8e}$$

$$u_1(x,y) \approx U_1(X,0) - KyU_0(X,0), \quad y \to \infty, \tag{2.8f}$$

$$p_1(x,y) \approx P_1(X,0) + Ky[U_0(X,0)]^2, \quad y \to \infty. \tag{2.8g}$$

Note that while the zeroth-order equations (2.6) are non-linear, the first-order equations (2.7) are linear and hence allow superposition (as first noted by Rott & Lenard (1959)). It is thus possible to consider separately the displacement effect (which gives rise to the terms $U_1(X,0)$ and $P_1(X,0)$ in (2.8f) and (2.8g)), and the curvature effect (which gives the terms multiplied by K in (2.7) and (2.8)), and superpose them in the end. Here, we are primarily interested in the curvature effects, so we ignore P_1 and all displacement effects. Eliminating p_1 with the help of (2.7b) and (2.8g), the two momentum equations (2.7b) and (2.7c) can be combined to obtain

$$u_0\frac{\partial u_1}{\partial x}+v_0\frac{\partial u_1}{\partial y}+u_1\frac{\partial u_0}{\partial x}+v_1\frac{\partial u_0}{\partial y}$$
$$=-\frac{\partial}{\partial x}\left\{K\int_0^y u_0^2\,dy+K\int_0^\infty (U_{0s}^2-u_0^2)\,dy\right\}+\frac{\partial^2 u_1}{\partial y^2}$$
$$+K\left\{y\left(u_0\frac{\partial u_0}{\partial x}+\frac{\partial p_0}{\partial x}\right)+\frac{\partial u_0}{\partial y}-u_0v_0\right\}, \quad (2.9a)$$

where $U_{0s}=U_0(X,0)$ is the surface speed from the outer solution. The continuity equation (2.7a) is

$$\frac{\partial u_1}{\partial x}+\frac{\partial}{\partial y}(v_1+Kyv_0)=0; \quad (2.9b)$$

and the boundary conditions to go with the above equations are, from (2.8e) and (2.8f),

$$u_1=v_1=0, \quad \text{at} \quad y=0, \quad (2.10a)$$

and

$$u_1(x,y)=-KyU_{0s}, \quad \text{as} \quad y\to\infty. \quad (2.10b)$$

The system of relations (2.9) and (2.10) constitute the governing equations for first-order boundary-layer flows allowing for longitudinal curvature effects.

3. Similarity analysis

In order to study some specific cases in detail and obtain a few standard solutions, we now seek the conditions under which similarity solutions for the system (2.9) and (2.10) exist. Similarity in the higher order system implies similarity also for the ordinary boundary-layer system (2.6), which leads to the well-known Falkner–Skan flows. One way of obtaining these is by introducing the transformations

$$\left.\begin{aligned}\xi=\int_0^x U_{0s}\,dx, \quad \eta=(2\xi)^{-\frac{1}{2}}U_{0s}y,\\ \psi_0(x,y)=(2\xi)^{\frac{1}{2}}f_0(\eta),\end{aligned}\right\} \quad (3.1)$$

where ψ_0 is the zeroth inner stream function and f_0 is a function only of η and not of ξ. When these variables are substituted into (2.6), it will be found that one can get an ordinary differential equation for f only if

$$\beta\equiv 2\frac{d\ln U_{0s}}{d\ln\xi}=\text{const.}, \quad U_{0s}=Cx^m, \quad m\equiv\beta/(2-\beta), \quad (3.2)$$

C being an arbitrary constant, and we obtain for f_0 the well-known Falkner–Skan equation

$$f_0'''+f_0f_0''=\beta(f_0'^2-1), \quad (3.3)$$

where primes denote differentiation with respect to η.

For the higher order system, we see from the continuity equation (2.9*b*) that a function ψ_1 can be defined such that

$$u_1 = \frac{\partial \psi_1}{\partial y}, \quad v_1 + K v_0 y = -\frac{\partial \psi_1}{\partial x}, \tag{3.4}$$

and in analogy with the lower order similarity, we require that

$$\psi_1(x,y) = (2\xi)^{\frac{1}{2}} f_1(\eta). \tag{3.5}$$

It is easily shown that similarity is possible only if

$$K = \frac{kU_{0s}}{(2\xi)^{\frac{1}{2}}} = k\left[\frac{C(m+1)}{2}\right]^{\frac{1}{2}} x^{\frac{1}{2}(m-1)}, \tag{3.6}$$

where k is a constant. It may be pointed out here that there is not necessarily a contradiction in prescribing $K(X)$ and $U_0(X,0)$ independently, because the latter is not uniquely determined by the surface under consideration. There is the possibility that the velocity distributions (3.2) can be obtained on the surfaces (3.6) by placing suitably other bodies or surfaces in the flow. In any case, our purpose in solving for similarity flows is to have a standard set of solutions on the basis of which approximate methods of solution for arbitrary curvatures and pressure gradients can be constructed.

The surfaces defined by (3.6) are identical with those obtained by Murphy (1953), though his analysis is different from ours. This is not surprising for the condition for similarity is simply that the new parameter introduced into the problem by a consideration of surface curvature, namely the quantity $K\delta$, should be independent of x for each of the Falkner–Skan flows. This directly leads to the requirement (3.6).

Murphy (1953) has given sketches of some of the surfaces $K \sim x^{-\frac{1}{2}}$, corresponding to $m = 0$ in (3.6). For arbitrary values of m, (3.6) is easily integrated in terms of the intrinsic variables of the surface; and its Cartesian co-ordinates s_1, s_2 can be given in the parametric form

$$s_{1,2} = \left(\frac{2}{m+1}\right)^{m/(m+1)} (k^2 C)^{-1/(m+1)} I_{1,2}\left[k\left(\frac{2C}{m+1}\right)^{\frac{1}{2}} x^{\frac{1}{2}(m+1)}, \frac{1-m}{1+m}\right],$$

where

$$I_{1,2}(z,\alpha) \equiv \int_0^z (\cos t, \sin t)\, t^\alpha\, dt$$

can always be reduced, by integration by parts, to the generalized sine and cosine integrals of Kreyszig (1953).

It is clear that for $m < 1$ the curvature at the leading edge is infinite, but this still does not contradict the basic assumption of 'moderate' curvature, which in non-dimensional terms means that the *product* $K\delta$ (or $k\epsilon$) is small. For all these surfaces δ goes to zero sufficiently fast near the leading edge for $K\delta$ to be a constant, hence it is perfectly consistent to assume it small. However, the present analysis does not take account of the so-called leading-edge effect.

Assuming that the curvature is given by (3.6), the system (2.9) can be reduced, with the help of (3.1–6), to the following single differential equation for f_1:

$$f_1''' + f_0 f_1'' - 2\beta f_0' f_1' + f_0'' f_1 = k\left[f_0''(\eta f_0 - 1) - f_0 f_0' - \beta\left\{\eta(f_0'^2 - 1) - \frac{2}{1+\beta}(f_0'' + f_0 f_0' + \beta\eta - A)\right\}\right]. \tag{3.7}$$

Here
$$A \equiv \lim_{\eta \to \infty} (\eta - f_0) \tag{3.8}$$
and depends on β. The boundary conditions given by (2.10) can be expressed in similarity variables as
$$f_1(0) = 0 = f_1'(0), \quad f_1'(\eta) \approx -k\eta \quad \text{as} \quad \eta \to \infty. \tag{3.9}$$

4. Solution of the differential equations

Equation (3.7) has been solved numerically by Cooke (1966) and by us independently. We programmed the problem for the Sirius computer at the National Aeronautical Laboratory, Bangalore, using a Runge–Kutta procedure with Gill's improvement. The basic Falkner–Skan solutions were taken from the tables of Smith (1954), and a Lagrangian interpolation scheme with six points

η \ β	−0·19	−0·16	−0·14	−0·10	0	0·4	1·0	2·0
0·0	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000
0·2	0·2733	0·2455	0·2527	0·2687	0·2987	0·3477	0·3673	0·3733
0·4	0·5918	0·5293	0·5403	0·5664	0·6148	0·6837	0·6970	0·6820
0·6	0·9572	0·8520	0·8625	0·8916	0·9447	1·0012	0·9837	0·9289
0·8	1·3690	1·2114	1·2164	1·2400	1·2821	1·2936	1·2276	1·1279
1·0	1·8240	1·6025	1·5959	1·6044	1·6184	1·5560	1·4339	1·2957
1·2	2·3149	2·0164	1·9915	1·9740	1·9430	1·7861	1·6102	1·4477
1·4	2·8303	2·4404	2·3899	2·3359	2·2450	1·9843	1·7659	1·5953
1·6	3·3533	2·8581	2·7751	2·6753	2·5144	2·1539	1·9100	1·7462
1·8	3·8628	3·2508	3·1300	2·9783	2·7442	2·3010	2·0506	1·9044
2·0	4·3343	3·5998	3·4386	3·2337	2·9316	2·4332	2·1938	2·0714
2·2	4·7429	3·8890	3·6886	3·4351	3·0788	2·5585	2·3436	2·2468
2·4	5·0666	4·1513	3·8744	3·5827	3·1930	2·6841	2·5020	2·4295
2·6	5·2902	4·2544	3·9976	3·6834	3·2844	2·8158	2·6696	2·6179
2·8	5·4090	4·3345	4·0678	3·7493	3·3651	2·9571	2·8009	2·8104
3·0	5·4300	4·3625	4·1003	3·7956	3·4464	3·1096	3·0287	3·0058
3·2	5·3718	4·3577	4·1135	3·8377	3·5374	3·2734	3·2173	3·2031
3·4	5·2611	4·3412	4·1255	3·8889	3·6441	3·4471	3·4100	3·4016
3·6	5·1285	4·3321	4·1511	3·9587	3·7690	3·6290	3·6056	3·6009
3·8	5·0024	4·3449	4·2006	4·0517	3·9121	3·8171	3·8030	3·8003
4·0	4·9060	4·3880	4·2786	4·1691	4·0711	4·0097	4·0015	4·0001
4·2	4·8536	4·4645	4·3855	4·3088	4·2433	4·2053	4·2008	4·2000
4·4	4·8517	4·5726	4·5182	4·4669	4·4252	4·4027	4·4003	—
4·6	4·8990	4·7078	4·6721	4·6395	4·6141	4·6013	4·6001	—
4·8	4·9896	4·8645	4·8421	4·8223	4·8076	4·8007	4.7999	—
5·0	5·1151	5·0370	5·0236	5·0120	5·0039	5·0003	5· 0000	—
5·2	5·2670	5·2204	5·2127	5·2063	5·2019	5·2000	—	—
5·4	5·4375	5·4107	5·4066	5·4032	5·4009	—	—	—
5·6	5·6120	5·6054	5·6033	5·6015	5·6004	—	—	—
5·8	5·8102	5·8025	5·8014	5·8006	5·8000	—	—	—
6·0	6·0050	6·0011	6·0006	6·0002	—	—	—	—
6·2	6·2024	6·2004	6·2002	6·2000	—	—	—	—
6·4	6·4011	6·4000	6·4000	—	—	—	—	—
6·6	6·6006	—	—	—	—	—	—	—
6·8	6·8000	—	—	—	—	—	—	—

TABLE 1. Values of $-f_1'(\eta)/k$ for selected values of the Falkner–Skan parameter β

was employed to obtain intermediate values at intervals of $\Delta\eta = 0{\cdot}05$. To satisfy the boundary conditions (3.9), two of which are given at $\eta = 0$ and the third at $\eta = \infty$, we computed two different solutions of (3.7) assuming arbitrary values for $f_1''(0)$. Neither of these solutions will in general satisfy the boundary condition at $\eta = \infty$, but one can always find a suitable linear combination which does so, this being admissible as (3.7) is linear.

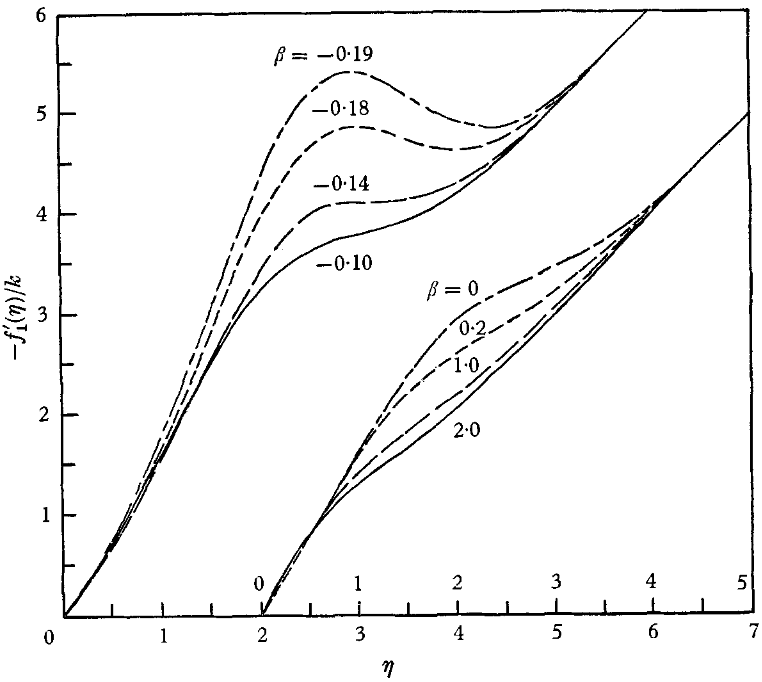

FIGURE 1. Change in velocity profile due to surface curvature for selected values of β. $u/U_{0s} = f_0'(\eta) + \epsilon f_1'(\eta)$.

The solutions so obtained are presented† in table 1 as $f_1'(\eta)$ *vs.* η for certain selected values of β; figure 1 shows a few typical velocity profiles. The results are believed to be correct to four figures, but errors of a few units in the fourth place are possible.

For all values of β for which equation (3.7) was solved, the effect of the curvature on the skin friction, displacement thickness and momentum thickness was also computed; the results are given in table 2 in the form of certain coefficients defined as follows. The local skin friction coefficient and Reynolds number are, respectively,

$$c_f = \tau_0/\tfrac{1}{2}\rho(U_{0s}U^*)^2, \quad R_x = U_{0s}x/\epsilon^2,$$

where τ_0 is the wall shear stress. The increment in c_f due to curvature is then given by

$$R_x^{\frac{1}{2}}\Delta c_f = [2(m+1)]^{\frac{1}{2}}\epsilon f_1''(0). \tag{4.1}$$

† More detailed tables, and an approximate method for solving the equations, will be found in Report AE 164A of the Department of Aeronautical Engineering, Indian Institute of Science.

We define the non-dimensional displacement and momentum thicknesses by

$$\delta^* = \frac{\epsilon}{U_{0s}}\int_0^\infty [U(Y)-u(y)]\,dy = \epsilon\delta_0^* + \epsilon^2\delta_1^* + O(\epsilon^3),$$

$$\theta = \frac{\epsilon}{U_{0s}^2}\int_0^\infty u(y)[U(Y)-u(y)]\,dy = \epsilon\theta_0 + \epsilon^2\theta_1 + O(\epsilon^3),$$

β	m	$\frac{f_1''(0)}{k}$	$-\frac{R_x^{\frac{1}{2}}\Delta c_f}{k\epsilon}$	$\frac{U_{0s}}{(2\xi)^{\frac{1}{2}}}\frac{\delta_1^*}{k}$	$\frac{U_{0s}}{(2\xi)^{\frac{1}{2}}}\frac{\theta_1}{k}$
−0·1988	−0·09042	6·244	8·416	34·11	−17·52
−0·195	−0·08884	1·5714	2·1212	8·922	−4·840
−0·19	−0·08676	1·2572	1·6991	6·635	−3·581
−0·185	−0·08466	1·1477	1·5529	5·603	−2·968
−0·18	−0·08257	1·1343	1·5365	5·125	−2·651
−0·16	−0·07407	1·1340	1·5433	4·002	−1·840
−0·14	−0·06542	1·1782	1·6108	3·452	−1·419
−0·10	−0·04762	1·2715	1·7548	2·837	−0·843
−0·05	−0·02439	1·3691	1·9124	2·398	−0·388
0	0	1·4469	2·0462	2·109	−0·096
0·1	0·05263	1·5630	2·2679	1·731	0·268
0·2	0·11111	1·6456	2·4531	1·486	0·476
0·4	0·25	1·7561	2·7766	1·173	0·681
0·6	0·42857	1·8315	3·0958	0·987	0·757
0·8	0·66667	1·8766	3·4261	0·839	0·776
1·0	1·0	1·9132	3·8264	0·737	0·772
1·2	1·5	1·9414	4·3412	0·657	0·752
1·6	4·0	1·9819	6·2674	0·542	0·702
2·0	∞	2·0092	∞	0·461	0·662

TABLE 2. Change in skin friction coefficient, displacement thickness and momentum thickness due to curvature

where it is easily shown that

$$\frac{U_{0s}}{k(2\xi)^{\frac{1}{2}}}\delta_1^* = -\int_0^\infty (\eta + f_1')\,d\eta, \tag{4.2a}$$

$$\frac{U_{0s}}{k(2\xi)^{\frac{1}{2}}}\theta_1 = \int_0^\infty [f_1'(1-f_0') - f_0'(\eta + f_1')]\,d\eta. \tag{4.2b}$$

Table 2 lists values of the quantities (4.1), (4.2a) and (4.2b); figure 2 shows the skin friction data graphically.

For those values of β for which computations have been made both by us and by Cooke, the values of $f_1''(0)$ differ at the most by one unit in the last place. For $\beta = 1$, Van Dyke (1964) gives $f_1''(0)/k = -1{\cdot}91$, in agreement with our value of $-1{\cdot}9132$. A striking feature of figure 2 is that as β approaches the critical value ($= -0{\cdot}198838\ldots$ according to Smith) at which the Falkner–Skan profile shows separation, $|f_1''(0)|$ seems to increase very rapidly. This suggests that curvature of the surface may have a strong influence on separation.

Before the computer results were available, an approximate calculation made for the case $\beta = 0$ (Ojha 1964), using the method of Meksyn (1961), gave $f_1''(0)/k \simeq -1{\cdot}44$, in good agreement with the computer result.

For all values of β, these results show a decrease in skin friction on the convex side of the surface, and confirm a trend first predicted by Murphy (1953). However, there are numerical differences; and a critical assessment of Murphy's and other previous work, together with a comparison with present results and analysis, is given in the next section.

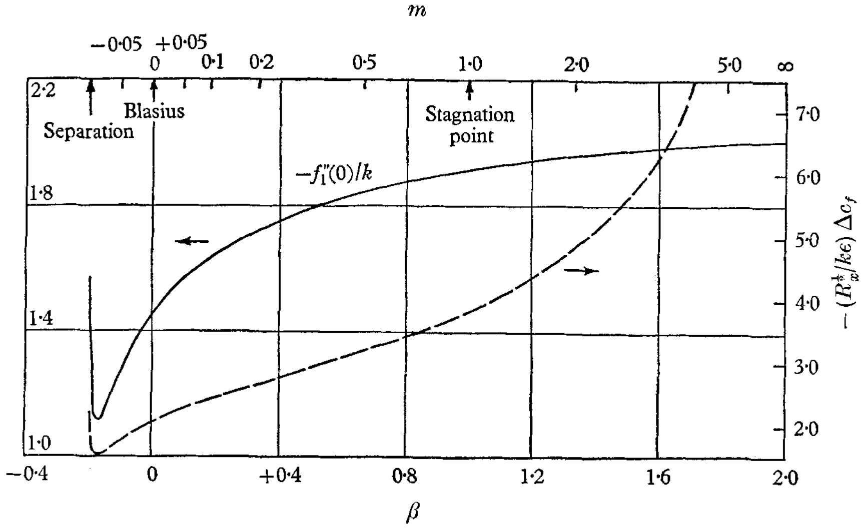

FIGURE 2. Change in skin friction due to surface curvature as a function of the pressure gradient parameter β. The curve is faired through the points listed in table 2.

5. Discussion

5.1. *Comparison with previous work*

The dependence of the skin friction on the surface curvature parameter has been studied previously by various workers. Their results for the case of flow without any pressure gradient are presented as $c_f R_x^{\frac{1}{2}}$ *vs.* $k\epsilon$ in figure 3, and the slopes of these curves at the origin are listed in table 3. Even for this single case, there are many discrepancies, some of which have been attributed to inadequate numerical computation; but the present work indicates that there are deeper reasons involving the formulation of the problem. The following brief remarks on each of the analyses indicate the reasons for the observed discrepancies.

Murphy (1953, 1965) uses the same co-ordinate system as we do. In his earlier work, he first formulates the problem for 'large' curvatures (i.e. $k\epsilon = O(1)$), although ultimately he retains only first-order terms in a parameter A which is proportional to our $k\epsilon$. It is important to note that for large curvatures, the surface will appear sharp to the outer flow; hence the inviscid surface velocity will in general tend to infinity in the limit—a fact which does not seem to have been considered by Murphy in writing his boundary conditions. Murphy in the same paper (1953) obtains a simplified set of equations stated to be valid only for moderate curvature. A comparison of these equations with those obtained by us here shows that while he includes a higher order term in the second momentum equation, he has not included similar higher order terms in the continuity and

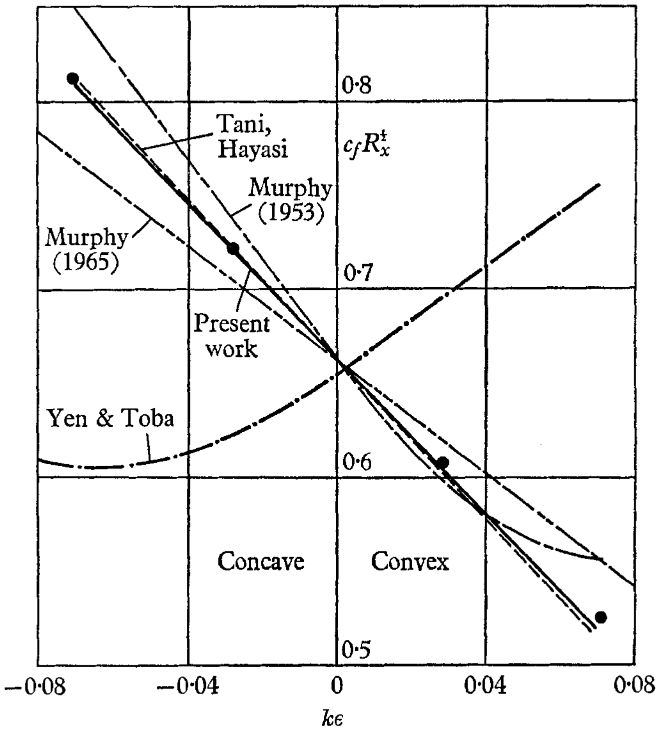

FIGURE 3. Comparison of previous work with present results for the effect of surface curvature on skin friction in zero pressure gradient. The symbol ● refers to the results of Schultz-Grunow & Breuer (1965).

Source	$-\left.\dfrac{d(c_f R_x^{\frac{1}{2}})}{d(k\epsilon)}\right\vert_{\epsilon=0}$
Murphy (1953)	2·48
Tani (1954)	2·05
Yen & Toba (1961)	−1·38
Hayasi (1963)	2·05
Murphy (1965)	1·41
Schultz-Grunow & Breuer (1965)	2·04
Present work	2·045

TABLE 3. Rate of change of skin friction with surface curvature in the absence of pressure gradient, as given by various workers

first momentum equations. Thus, even his analysis for moderate curvature does not appear to be consistent to the order considered. Unfortunately, his later work (Murphy 1965) is still open to similar objections. Again, he does not split his solutions into zeroth and higher order terms as we do, although his computations are made only for small values of a parameter Ω, equal to our $k\epsilon$. If his dependent variables are expanded as a series in Ω, we obtain to $O(1)$ the classical boundary-layer equations (2.6); the next set of equations, to $O(\Omega)$, show, on comparison with ours, that he omits the right-hand side in the continuity equation (2.7a), and the terms proportional to Ky in (2.7c). All these terms are $O(\epsilon)$ or $O(\Omega)$, and hence not negligible in comparison with the other terms of (2.7) which are of the same order and are included by Murphy. A similar analysis of Murphy's final similarity equation reveals that certain terms proportional to η are missing. For example, in the simplest case of $\beta = 0$, the term $k\eta f_0 f_0''$ on the right of (3.7) does not appear in Murphy's equation, though it is clearly of the same order as the other terms.

Yen & Toba (1961) adopt a different co-ordinate system, consisting of streamlines and their orthogonal trajectories. In contrast with the trend of our and Murphy's results, they find an increase of skin friction on the convex side. (See also Murphy 1962; Yen & Toba 1962.) The discrepancy seems again to be due at least in part to inconsistent approximations. Their analysis involves, in addition to the equation of motion, the Gauss equation connecting the curvatures of the co-ordinate lines. While the first higher order terms seem to have been included in the equations of motion, Yen & Toba's simplified Gauss equation omits terms of the same order, leading again to corresponding inconsistencies in the final equation.

Hayasi (1963) attributes the differences in the results of Murphy and of Yen & Toba to inadequate numerical computation. He based his own work on Yen & Toba's equations, and so the above remarks apply equally to his work.

As we stated earlier, any more general analysis valid also for larger curvatures should reduce to the correct equations also in the limit of moderate curvature (i.e. $K\delta \rightarrow 0$). But as the analyses mentioned above do not consistently include all terms of the same order even in the limit, they would seem to be deficient at large curvatures as they are at moderate curvatures.

Another recent contribution to the subject is that of Schultz-Grunow & Breuer (1965), who formulate a non-linear equation to describe the effects of curvature. Their equations contain all the terms of $O(1)$ and $O(\epsilon)$ present in Van Dyke's and our equations, and include in addition (only) one viscous term of $O(\epsilon^2)$; but the exact metric factor is preserved in all the terms kept. They further assume that the potential outer flow has no cross velocity (i.e. $V = 0$), which is permissible to $O(\epsilon)$ as the flow can then be split into curvature and displacement effects (see § 2). Thus, their formulation is certainly correct to $O(\epsilon)$ and hence is an improvement over Murphy's work, though to this order it is unnecessarily complicated; its validity to higher orders is questionable, because of the assumptions mentioned above. They only consider the flow with no pressure gradient; their value for $d(c_f R_x^{\frac{1}{2}})/d(k\epsilon)$, deduced by us from their data for $k = \pm 2^{\frac{1}{2}}$ (0·02), is in good agreement with our results.

Rather similar comments apply to the work of Massey & Clayton (1965), who also construct a non-linear equation which agrees with ours to the two lowest orders; their value for $d(c_f R_x^{\frac{1}{2}})/d(k\epsilon)$ also shows close agreement. They include in their analysis a consideration of the displacement effect by a procedure which amounts (in the terminology of this work) to matching the outer limit of the inner solution, $u_1(y \to \infty)$, with $U_0(Y = \delta^*)$. This is clearly an oversimplification.†

It is clear from the above that all the theories mentioned here are strictly entitled only to discuss the slope $d(c_f R_x^{\frac{1}{2}})/d(k\epsilon)$ at the origin. There seems to be no justification for attaching any significance to the curvature of the curves shown in figure 3.

The work of Tani (1954), on the other hand, stands out as a sound analysis. Tani considered only the flow without pressure gradient, and used a small perturbation scheme. Though he did not use the method of matched asymptotic expansions, he obtained the correct equations and boundary conditions, and his final result for the skin friction coefficient (see table 3), agrees well with our result. The present formulation is, however, thought to be more systematic, and considers also flows with pressure gradients.

5.2. *Concluding remarks*

Like any similarity solution, the present analysis applies only to a special family of surfaces and pressure distributions. To tackle the problem when these are arbitrarily given, a Pohlhausen-type procedure to solve the higher order equations (2.9) can be adopted. Some preliminary calculations using this technique confirm the trend of the results obtained by the similarity analysis; these will be reported in the near future.

In conclusion, two comments need to be made. First, the reduction in skin friction due to convex curvature implies that separation is hastened. Present results show that even in the absence of a pressure gradient, the skin friction vanishes if $k\epsilon = 0{\cdot}328$, or equivalently if $K\delta^* = 0{\cdot}40$, using the Blasius value for the displacement thickness δ^*. Of course it is not to be expected that the present analysis would be valid right up to separation, but it is interesting that the $K\delta^*$ required for separation is not large; it certainly seems suggestive.

Secondly, it must be emphasized that the effect of curvature on laminar flows in general will depend on the constraints on the flow (or in other words on the outer boundary condition). An examination of the sign of the terms appearing in the higher equations (e.g. (3.7) with $\beta = 0$) suggests that the dominant reason for the reduction of skin friction on the convex side is that in the irrotational outer flow, the velocity tends to decrease away from the surface. Where this effect is absent, the change in skin friction may be quite different. We may cite the example of cylindrical Couette flow, with inner cylinder at rest and outer cylinder rotating. The skin friction on the convex surface of the inner cylinder is in this case more than in plane Couette flow with same gap width and velocity difference.

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† Incidentally, there do not seem to be any strict similarity solutions including displacement effects. It is easily shown that for the Blasius solution the displacement speed $U_1(X, 0)$ is zero on a flat plate. However, as this speed is to be found by integrating the contributions from a source distribution of strength $V_1(X, 0)$ along the surface, it is clear that $U_1(X, 0)$ will depend on the geometry of the surface and will not necessarily be zero on the curved surface with zero pressure gradient in our similarity solutions.

made available to us the results of some of his unpublished computations, and we acknowledge a useful correspondence with him. Mr S. S. Krishnamurthy has been of considerable assistance to us in the computations.

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